Estimation of parameters of under-damped second order plus
death time processes using relay feedback

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Abstract

Unknown process parameters are identified using limit cycle data of the relay–response curves obtained from single relay feedback test on second order plus dead time (SOPDT) processes. Time domain analytical expressions of these curves are helpful in deriving conditions to estimate process parameters accurately. Identification algorithms are formulated to estimate minimal number of model parameters of SOPDT processes. The estimation procedure consists of the following steps. A relay feedback test is conducted to obtain a symmetric relay response (if the response is not symmetric we do a biased relay test) and then estimation algorithms are applied. The method is validated for systems with modeling errors or under load and also for systems with and without measurement noise. © 2006 Elsevier Ltd. All rights reserved.

Keywords: Relay feedback; Modeling; Identification

1. Introduction

Single relay feedback is mostly used to identify process parameters because of its simplicity. It is used to find transient dynamics of the process and thereby to tune the feedback controller. The reliability of the method is made used of by Astrom and Hagglund (1984) for PID controllers. Process identification using autotuning variation (ATV) is one useful tool today in process industries. Luyben (1987) is one of the pioneers to explain ATV technique for system identification of low order processes in which ultimate gain and ultimate frequency are obtained (Astrom’s autotuning method). Process model parameters were obtained from relay response of the 2nd order system (Huang et al., 2005). But these methods use frequency domain parameters ($K_u$ and $P_u$), which are derived from describing functions and carry only approximate information of process at ultimate frequency. Few articles have been reported to find exact parameter estimation of low order transfer functions using single relay test (Tan, Wang, & Lee, 1998; Wang, Hang, & Zou, 1997) or by using A-locus method (Kaya & Atherton, 2001; Majhi & Atherton, 1999). But closed-form solutions are not available for general transfer functions. Progresses in relay feedback up to 1999 is described elaborately by Yu (1999). Most of the literature discusses identification of process dynamics using initial part of the relay feedback (RFB) responses. But in practice it is difficult to store the very initial part of RFB response. So, effort has been made in this work to extract unknown process parameters from stabilized part of the RFB response. Moreover, estimation techniques using frequency domain ultimate properties alone do not yield very accurate process parameters and hence tuning gives a poor controller performance. Therefore, there is a need to identify exact parameters of the process. Mathematical models of relay feedback responses in time domain are helpful to estimate exact process parameters. Recently, Luyben (2001) discussed a very simple technique that needs one additional parameter (other than limit cycle data), namely, shape factor, to identify process transfer functions using a single relay feedback test for stable and unstable FOPDT systems. The shapes of relay response curves primarily give an idea of the system category and its order. Thyagarajan and Yu (2002) categorized process models by observing the shapes of relay feedback response (generated from mostly FOPDT processes with different $D/\tau$ ratio and higher order systems) and identified the transfer function models.

The purpose of this work is to identify process model parameters for under-damped SOPDT process from limit cycle data.
to calculate appropriate PID controller settings. This paper is organized as follows: an introduction to relay feedback is presented in Section 2. The identification of model parameters is explained in Section 3. Concluding remarks are drawn at the end. Proposed scheme of identification is shown in Fig. 1.

2. Relay feedback response

2.1. Relay feedback

Astrom and Hagglund (1984) proposed that if a relay of magnitude ‘h’ is inserted in a feedback loop, the input a(t) becomes ‘h’. If the relay output lags behind the input by π radians, the closed-loop system starts oscillating around set-point with a period of \( P_u \). As the output \( y(t) \) starts increasing, the relay output switches to opposite direction and becomes \( a(t) = -h \). With a phase lag of π, a limit cycle of amplitude ‘a’ is formed and the process variable crosses the set-point. From the principle harmonic approximation of the oscillations, the ultimate gain \( K_u \) can be approximated (Astrom & Hagglund, 1984) as \( K_u = 4h/\pi a \), and ultimate frequency \( \omega_u \) thus becomes \( \omega_u = 2\pi P_u \) (where \( P_u \) is the period of oscillation). With the help of these two identified frequency domain parameters, \( K_u \) and \( P_u \), many tuning rules can be formulated. We consider here under-damped SOPDT systems as they are rich in process dynamics. Note that all the process gains are assumed to be 1 and input relay height used is of \( h = 1 \).

Systems with under-damped oscillations has the following model structure

\[
G_p(s) = \frac{K_p e^{-t_s}}{1 + 2\xi \omega_s^2 s + \omega_s^4}
\]

(1)

where \( \xi \) is the damping factor with \( \xi < 1 \).

3. Identification of transfer functions

After doing the relay feedback test, the next work is to identify the unknown system parameters for that process model. The procedures for identification are explained below. (In this work, relay height, \( h \) is taken as unity.)

SOPDT systems with low damping coefficient, \( \xi \), and moderate to high \( D/\tau \) ratio are considered in this case. This type of processes has four unknown parameters, namely, \( K_p \), \( \tau \), \( \xi \) and \( D \), as it appears in Eq. (1). The analytical expression for the relay feedback response of this kind of processes is given as (Panda & Yu, 2003):

\[
y(t) = k_p h \left( \frac{1 - 2^{-\beta t/\tau}}{\beta \sin \left( \frac{\beta \tau}{\tau} + \alpha \right)} \right)
\]

(2)

where

\[
\alpha = \frac{\beta (1 + 2r \cos(\theta)) - 2r \sin(\theta)}{\xi + \beta \cos(\theta) + 2r \sin(\theta)}
\]

\[
r = e^{-P_u \beta t / 2} \quad \text{and} \quad \theta = \frac{P_u \beta t}{2}
\]

With assumptions of ideal and symmetric relay responses the following boundary conditions are framed for Eq. (2) that can be used to estimate above mentioned four unknown parameters

\[
\begin{align*}
\gamma_{t = P_u/2} & = 2 \\
\frac{dv}{dt} & = 0 \\
\gamma_{t = P_u/2} + \gamma_{t = P_u/2} & = 0 \\
\gamma_{t = P_u/2} - D & = 0
\end{align*}
\]

(4) \hspace{1cm} (5) \hspace{1cm} (6) \hspace{1cm} (7)

The relay feed-back response (Fig. 2) of underdamped SOPDT systems (with high \( D/\tau \) value) has more than one peak per cycle (easy to locate landmark points). The landmarks are starting point (\( t = 0 \)), ending point (\( t = P_u/2 \)), \( t_{peak} \) and zero-crossing of response. If one can locate point A or B (as in Fig. 2) on the relay response obtained from experiment, then it becomes easy to find equations to calculate unknown parameters. But the
real problem is to find out starting and ending points of response. We assume the following to formulate the Eqs. (4)–(7).

(i) The relay responses are symmetric about zero baseline.
(ii) The response of processes that shows more than one peak per cycle (second scenario) starts from point A and ends at point B (as shown in Fig. 2).
(iii) For the first scenario, the response starts from bottom most point and ends at next top point (as in Fig. 3).
(iv) If more than one peak exists, we consider the last peak for calculation of $D^*$, where $D^*$ is apparent dead time.

Now $t_{peak}$ can be written as $t_{peak} = D^* - D + P_u/2$. Substituting Eq. (4) in Eq. (2), we get

$$k_p = \left\{ 1 - \frac{2}{P_u} e^{-P_u/2\tau} \left[ \sin \left( \frac{P_u}{2\tau} + \alpha \right) \right] \right\} = \alpha \quad (8)$$

Condition in Eq. (5) gives

$$\tau \left[ \pi - \alpha \right] - t_{peak} = 0 \quad (9)$$

Replacing Eq. (6) in Eq. (2) yields

$$\frac{1}{\tau} \left[ \sin(\alpha) + e^{-\xi P_u/2} \sin \left( \frac{P_u}{2\tau} + \alpha \right) \right] = 1 \quad (10)$$

and according to Eq. (7) one can obtain from Eq. (2)

$$k_p = \left\{ 1 - \frac{2}{P_u} e^{-\xi (0.5P_u - D)/\tau} \left[ \sin \left( \frac{\beta (0.5P_u - D) + \alpha}{\tau} + \alpha \right) \right] \right\} = 0 \quad (11)$$

These equations can be solved simultaneously to find $K_p$, $D$, $\tau$, and $\xi$.

Alternatively, a simple procedure can be adopted. The dead time ($D$) and time needed to reach peak ($t_{peak}$) can be easily measured from relay response curve (see Fig. 2). Appropriate D can be found from initial part of the relay response curve. Time, '$t=0$' i.e. pt A of model starts from $D$ distance away along time axis from a point, when $y = 0$. Thus, if pt A is known, then $t_{peak}$ can be easily measured. Time from point $t = 0.5 P_u$ to point B gives a measure of dead time $D$. Height of point B (where $t = 0.5 P_u$) from zero baseline gives an estimate of $K_p$. Other two unknown parameters ($\tau$ and $\xi$) can be found separately from the above Eqs. (8) and (9).

Hence, the following steps may be followed to estimate the four unknown parameters of under-damped systems.

0. Estimate apparent dead time ($D^*$), i.e. time taken by actual relay response to reach its second peak. In case, there is more than one peak in a particular cycle of RFB response, we consider the last peak of that cycle for estimation of $D^*$.
1. Solve Eqs. (8)–(11) simultaneously to calculate $K_p$, $\tau$, $\xi$, and $D$.

This procedure enables us to find parameters for SOPDT under-damped type of models.

4. Disturbance and noises

4.1. Load disturbance

Load changes frequently occur in process industries. Disturbance rejection is a major criterion in chemical process control. Sensitivity with respect to load changes is an important consideration in evaluating identification techniques (Table 1).

Under load disturbance ($L = 0.5$ to a process model with transfer functions as given in Table 2), an ideal relay feedback test results in an asymmetric oscillation (Fig. 4), and consequently, an imbalance in half periods results that leads to errors in estimates of $K_p$ and $\omega_u$. To overcome this load effect, a bias value ($\delta_0$) is added to the relay-input-height ($h_b$). Yu (1999) has shown that this bias value is related to amplitude of oscillation by
the processes are given in Table 2.

Following relation

\[ h_0 = -\frac{\Delta a}{h} \] (12)

where \( \Delta a \) is asymmetry in the output (if \( a_{\text{load}} \) is the amplitude of the system with load and \( a_{\text{normal}} \) is the amplitude of the system without load or bias, then \( \Delta a = a_{\text{load}} - a_{\text{normal}} \)) and \( a \) is the amplitude of output relay response. Thus, to restore the symmetry in output response, the relay is switched to an output bias so that the newly adjusted relay input height becomes \( h = h_0 + \delta \).

Let us take an example of critically damped system with transfer function, \( G = 10e^{-10}s/(s + 1)^2 \). Without load disturbance, the relay feedback test (with \( h = 0.8 \)) gives \( K_u = 2.5543 \) and \( \omega_u = 1.3159 \) and the response is shown in Fig. 5 (\( t = 0-20 \)). After introducing a load change of \( L = 0.02 \) at \( t = 20 \), an asymmetric sustained oscillation results (\( t = 20-40 \)). The estimated values of \( K_u \) and \( \omega_u \) become 2.5508 and 1.2955, respectively. From the system response we have \( \Delta a = -0.16 \) and \( a = 0.4 \). With the known values of \( \Delta a \) and \( a \), the bias value \( h_0 \) can be computed and the result becomes \( h_0 = 0.32 \). Next, an output biased relay feedback test is performed (\( t > 40 \)) (with the same load still active) and \( K_u \) and \( \omega_u \) are found to be 2.5542 and 1.3157, respectively, which are almost same that of disturbance free case. Hence, \( K_u \) and \( \omega_u \) values were regained. Therefore, output biased relay is very effective in identifying the quality of the process model in the face of load changes.

4.2. Presence of measurement noise

Measurement noise is a common problem in almost all process industries. It is necessary to know whether the shapes of relay feedback responses are deteriorated in the presence of noise or not. The proposed method for model identification was tested against measurement noise. One process is undertaken for study. That is

\[ G = \frac{10e^{-10}s}{s^2 + 1.6s + 1} \] (13)

Relay feedback tests (relay height = 1) were performed on these processes with noise. Noise to signal ratio (NSR) was 0 and 0.001. The limit cycle data were calculated by taking average of fictitious peaks (due to measurement noise in the relay response, many peaks are observed near actual peak. We calculate average of the neighborhood peaks near this actual peak for finding out amplitude) around the nominal peak of the stabilized response. Thus, amplitude (\( a' \)) and period of oscillation (\( P_{a'} \)) were found. The above mentioned identification algorithms were used to evaluate the model structures and are shown in Table 3. It can be seen that the calculated limit cycle data from noisy RFB response curves are almost closure to that of noise-free case. Hence identified model parameters are in close approximation to the process model.

As there are many tuning rules available (Panda et al., 2003), closed-loop controller tuning and performance studies are left out to readers.

5. Application

The present method of identification is used to estimate process model parameters of the following transfer function:

\[ G_p = \frac{10(3S + 1)e^{-0.5S}}{(S + 1)(5 + 1)} + \frac{10}{10}N(0, 1) \] (14)

where \( N \) is noise. This unknown process is identified (with relay height = 1.35) under NSR = 0 and NSR = 0.01 where by \( G_{in} \)
and \( G_m \) are obtained, respectively, whose parameter values are given in Table 3. Model mismatch can be calculated in terms of multiplicative error as

\[
\delta(j\omega) = \int_{\omega_u}^{\omega_u/\sqrt{10}} \left| \frac{G_m - G_p}{G_p} \right| d\omega
\]

The numerical value becomes 0.0025 with \( G_m = G_{m0} \), which reveals that there is model mismatch in low frequency regions. As there are many tuning rules (IMC, etc.) to compromise (by choosing \( \lambda \)) the model mismatch, closed-loop performance will no longer be a problem. The identified model is subjected to relay test to find out ultimate properties (\( K_u = 0.64 \) and \( \omega_u = 0.99 \)), which are found to be closer to those of process. The tolerance of the present method with measurement noise, is up to SNR = 5%.

The relay response is shown in Fig. 5. The method suggested by Rangaiah and Krishnaswamy (1996) to estimate parameters of SOPDT underdamped system on the same process, as in Eq. (14), \( G_p \), yielded \( G_{m0} = -3e^{-0.59}/(3.28 s^2 + 0.95 S + 1) \) (with point of infection, \( t_i = 3.05 \)). On comparing closed-loop simulation results on \( G_p \) with IMC-PID controllers tuned using \( G_{m0} \) and \( G_{m0}^{GR} \) (and \( \lambda = \tau_p \)) it is found that performances are quite satisfactory and IAE values for set point change are 4.49 (present method) and 5.4 (Rangaiah and Krishnaswamy method), respectively.

### Table 3

<table>
<thead>
<tr>
<th>True process</th>
<th>Identified process with NSR = 0</th>
<th>Identified process with NSR = 0.001 and NSR = 0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1.0 \times 10^{-10}(s^2 + 1.6 s + 1) )</td>
<td>( s^{-1} \times 10^{-10}(s^2 + 1.6 s + 1) ) ( a = 1.0303, \ P_u = 23.00 )</td>
<td>( 1.0111e^{-0.0001}(0.98 s^2 + 1.6 s + 1) ) ( a = 1.03186, \ P_u = 22.92 )</td>
</tr>
<tr>
<td>( -3(3 s + 1)e^{-t} ) ( s(3 s + 1) + a ) ( -10(3 s + 1) + N(0,1) )</td>
<td>( -2.991e^{-1.69}(3 s^2 + 1.68 s + 1) ) ( a = 2.993, \ P_u = 6.56 )</td>
<td>( -2.66e^{-1.69}(0.98 s^2 + 1.6 s + 1) ) ( a = 2.6663, \ P_u = 6.30 )</td>
</tr>
<tr>
<td>( -3(3 s + 1)e^{-t} ) ( s(3 s + 1) + a ) ( -10(3 s + 1) + N(0,1) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
6. Conclusion

This method of system identification (Fig. 1) is simple and will be of much industrial use (but with a limitation to transfer functions with numerator zeros). A single relay feedback test is carried out on unknown system. If the responses are symmetric (in case of asymmetric signals, we may have to do output bias relay adjustment for load or adjustment through hysteresis, at initial stage, for measurement noise) then limit cycle data will guide us to calculate the parameters of the unknown process. Analytical expressions for the relay response of these categories along with boundary conditions are helpful to identify the model parameters. The present method can be applied to higher order processes and also in presence of measurement noise or load disturbances.

References


